

Fluctuation-dissipation relations: achievements and misunderstandings

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We discuss the “generalized fluctuation-dissipation relations (theorems)” for the first time suggested by us in 1977-1984 as statistical-thermodynamical consequences of time symmetry (reversibility) of microscopic dynamics. It is shown, in particular, that our old results in essence contain, as alternative formulations or special cases, various similar relations (including the “fluctuation theorems”) what appeared after 1990.

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I. INTRODUCTION

During last fifteen years one can observe explosively growing interest in rigorous theoretical results of non-equilibrium statistical physics implied by time reversibility of microscopic motion. In part this is caused by appearance of new possibilities for justification of the theory on mesoscopic level. Several such experiments and some aspects of the underlying theory were reviewed one year ago in Physics-Uspekhi [1]. However, a state of the theory as the whole was reflected by [1] not quite correctly. The matter is as follows.

Interest in the mentioned rigorous results has very long history. It is sufficient to recollect the Kirchhoff law [2], Einstein relation [3, 5], Nyquist formula [4, 6] and the unifying fluctuation-dissipation theorem (FDT) [2, 7]. Later, the Efremov’s “quadratic FDT” [8] and Stratonovich’s “four-index relations” [9, 10] had appeared. We have displayed our interest in 1977-1981 in series of works where for the first time obtained [11] and investigated [12–15] the “generalized fluctuation-dissipation relations” (FDR), or theorems [14], which universally connect probabilities of observation of mutually time-reversed processes and entropy increments of a concerned system in these processes. The first of such relations was formula (7) from [11]. In the same works and in [16–22] we, and then in [23–28] one of us, considered also various modifications and applications of FDR.

Undoubtedly, many new representations and applications of FDR may arise with time. However, hardly one can rise the degree of generality of FDR as written in their abstract form, - e.g. like equality (2) from [17],

$$P(\Pi_+) \exp[-\Delta S(\Pi_+)] = P(\Pi_-) \quad (1)$$

(see paragraph III.B), - since such relations already absorbed all given by time reversibility of Hamiltonian micro-dynamics. Therefore, similar theoretical results of authors of the today’s “boom”, - started in 1997-1999 from works by Jarzynski [31–33] and Crooks [34–36], - add nothing to our FDR in the sense of generality.

Nevertheless, in current literature, even including reviews [37] (see also references in [29, 30, 38, 39]), the

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opinion propagates as if our results are particular cases of achievements of “new age”. At that, as a substantiation of this opinion, it is affirmed that we resorted to restrictive conditions, namely, coincidence of time-dependent Hamiltonian parameters at beginning and end of a time interval under consideration. Though it is easy to certain of falsity of this affirmation, merely thumbing through our papers (especially since main of them - [11–13] - are freely available from internet). To be precise, in review [38] and paper [39] (see also references therein) a different, rather objective, look at history of the subject is expressed. But it is not dominating, and, unfortunately, the misunderstanding had penetrated into review [1] too.

At this point, it is necessary to underline that our formulae and Jarzynski and Crooks formulae have different statistical meanings. At that, the first directly embrace not only closed systems, - to which in fact the second are addressed, - but also open systems. In spite of this essential difference, the first and the second outwardly resemble one another. Apparently, just their resemblance causes inadequate perception of our formulae by some authors and thus the mentioned misunderstandings. Therefore, their “victims” can see only a part of true contents of the theory. As the consequence, they do not see many applications of the theory and ways of its justification in experiments like that described in [1].

These circumstances induced us to write the present “methodical remarks”. Below, we shall try to prove, in visual fashion, all the aforesaid about relationship of our results and results of Jarzynski and Crooks (including the so-called “fluctuation theorems”) and about origin of the “misunderstandings”. Simultaneously, we shall point out key aspects of the generalized FDR, not pretending to review of all of their forms, but illustrating their possibilities by selected examples.

We hope that the material we suggest will be useful for all interested readers. For simplicity and brevity, we confine our consideration to classical statistical mechanics, partly following our critical notes [29, 30] and the review [1] which stimulated our activity (we are grateful to its author therefor). Specific remarks on quantum case can be found in [30].

II. OBSERVATIONS OF NON-EQUILIBRIUM SYSTEMS, THE LIOUVILLE THEOREM, AND STATISTICAL EQUALITIES

A. Objects under consideration

The matter concerns thermodynamically non-equilibrium dynamical systems subject to external influences. Speaking about evolution of such system from viewpoint of statistical mechanics, anyway one can not do without Hamiltonian of a system, $H(q, p; t)$, and probability distribution of its microscopic states, $D(q, p; t)$, where $\{q, p\} = \Gamma$ are system’s canonical variables. Time dependence of Hamiltonian can be

written also as $H(q, p, x(t))$ where $x(t)$ represent some external forces, or fields, or potentials, or governing parameters. At that, nowadays it is popular to term time dependence of $x(t)$ “protocol”.

Usually it is assumed that if a system stays left to his own for sufficiently long time, - i.e. external forces keep constant, $x(t) = x_0 = \text{const}$, and do not change its energy, - then one can prescribe for it the Gibbs thermodynamically equilibrium probability distribution function,

$$D_{eq}(\Gamma, x_0) = \exp \{ [F(x_0) - H(\Gamma, x_0)]/T \}, \quad (2)$$

where T is resulting temperature of the system, and $F(x_0)$ its free energy determined by the normalization condition: $\int D(\Gamma; t) d\Gamma = 1$. Obviously, this “axiom” presumes that the system either contains a large enough (ideally, infinite) “thermostat” or was contacted with such one.

B. The Jarzynski equality (JE) and related misunderstanding

Let us address to “misunderstandings”, pointed out in Introduction, as they have printed in [1]. More precisely, to comparison in [1] between “Jarzynski equality” from [31] (formula (9) in [1]) and “Bochkov-Kuzovlev equality” (evidently, formula (8) from [11]).

In [31], a system was considered which initially, at time moment $t = 0$, is in equilibrium state (2) and then evolves being led out of equilibrium by changing of external forces from $x(0) = x_0$ to $x(\theta)$ at some moment $\theta > 0$ when observation of the system is finished. (in [31] the forces are designated as $\lambda(t)$ while θ as t_f). The Jarzynski equality from [31] reads

$$\langle \exp(-W/T) \rangle = \exp(-\Delta F/T), \quad (3)$$

$$W = H(\Gamma(\theta), x(\theta)) - H(\Gamma, x(0)), \quad (4)$$

where W is change of full energy of the system during observation time interval; $\Delta F = F(x(\theta)) - F(x(0))$; $\Gamma(t)$ is current system’s microstate treated as strictly deterministic function of its initial microstate $\Gamma = \Gamma(0)$ and functional of preceding forces’ values; the angle brackets denote averaging over given distribution of initial microstates, $\langle \dots \rangle = \int \dots D(\Gamma; 0) d\Gamma$, in the present case - over $D_{eq}(\Gamma, x(0))$.

In this respect, in [1] it is said: “If final value of parameter λ coincides with initial one, $\lambda(t_f) = \lambda(0)$, i.e. the process is cyclic, then $\Delta F = 0$, and equality (9) reduces to that earlier obtained by Bochkov and Kuzovlev: $\langle \exp(-\beta W) \rangle = 1$ ”.

From this statement any reader must conclude, of course, that the Jarzynski equality is more general and, therefore, study of the theme should begin from [31].

However, if author of [1] reproduced derivations of both the equalities, not one of them only, then it would be clear that, firstly, they concern **different** observations (quantities) and even, generally speaking, different initial

distributions, so that their literal comparison (by their surface appearance) has no sense. Secondly, in [11] (as well as in other our works) there is nothing similar to the condition $\lambda(t_f) = \lambda(0)$ (moreover, time dependence of forces there has no restrictions at all).

Let us consider the actual state of affairs more carefully.

C. The Bochkov-Kuzovlev equality (BKE)

We have to emphasize two essential differences of our equality ((8) in [11]) from the Jarzynski equality (JE). First, in our equality the place of the quantity W is occupied by

$$E = H_0(\Gamma(\theta)) - H_0(\Gamma) , \quad (5)$$

while place of initial distribution (2) by

$$D(\Gamma; 0) = D_{in}(\Gamma) = \exp \{ [F_0 - H_0(\Gamma)]/T \} , \quad (6)$$

where time-independent function $H_0(\Gamma)$ is formally arbitrary (within reasonable limits, of course), and F_0 is determined by normalization (in [11] factor $\exp[F_0/T]$ was written as $1/q$). Second, our theory permits not only smooth time variations of the forces but also discontinuous variations (“jumps”).

Our equality reads

$$\langle \exp(-E/T) \rangle = 1 \quad (7)$$

It can be proved with the help of obvious identity

$$\exp(-E/T) D_{in}(\Gamma) = D_{in}(\Gamma(\theta)) \quad (8)$$

and the Liouville theorem, according to which Jacobian of micro-variables transformation from Γ to $\Gamma(t)$ always equals to unit, $d\Gamma/d\Gamma(t) = 1$ (phase volume is conserved). Namely,

$$\begin{aligned} \langle e^{-E/T} \rangle &= \int e^{-E/T} D_{in}(\Gamma) d\Gamma = \\ &= \int D_{in}(\Gamma(t)) \frac{d\Gamma}{d\Gamma(t)} d\Gamma(t) = 1 \end{aligned}$$

(since distribution $D_{in}(\Gamma)$ is normalized to unit).

Clearly, this derivation of BKE is completely indifferent to how the forces $x(t)$ behave with time. Replacing in these “calculations” $-E$ with $\Delta F - W$, and $D_{in}(\Gamma)$ with $D_{eq}(\Gamma, x(0))$, one gets proof of JE (3).

In [11] the function $H_0(\Gamma)$ was chosen in the form $H_0(\Gamma) = H(\Gamma, x_0)$ with definite fixed value x_0 (for instance, $x_0 = 0$), i.e. represents some essential part of full Hamiltonian and total energy of the system. In such case, the distribution (6) (generally non-equilibrium) coincides with equilibrium distribution (2).

However, in our consideration, in opposite to [31], not necessarily $x(0) = x_0$, that is forces can commit “jumps” in the beginning of observation, i.e. can be sharply “switched on” (as is natural to say at $x_0 = 0$).

D. Comparison between JE and BKE

In view of the mentioned possible arbitrariness in choice of $H_0(\Gamma)$, BKE (7) can not be deduced from JE (3), while the opposite is easy realizable, just due to the arbitrariness [29]. Thus, with regard to formal generality the priority belongs to BKE. Nevertheless, after a choice of $H_0(\Gamma)$ is made, $H_0(\Gamma) = H(\Gamma, x_0)$, these equalities become independent. It is useful to compare them at $x(0) = x_0$, when both they correspond to exactly same protocols and statistical ensembles.

Let us write system’s Hamiltonian in the form

$$H(\Gamma, x) = H_0(\Gamma) - h(\Gamma, x) , \quad (9)$$

$$h(\Gamma, x) = H(\Gamma, x_0) - H(\Gamma, x) , \quad (10)$$

interpreting $H_0(\Gamma)$ as “internal energy” of the system, while $-h(\Gamma, x)$ as its “energy of interaction” with sources of the forces. Then, according to system’s equations of motion (Hamilton equations),

$$W = - \int_0^\theta \frac{dx(t)}{dt} \cdot \frac{\partial h(\Gamma(t), x(t))}{\partial x(t)} dt , \quad (11)$$

$$E = \int_0^\theta \frac{d\Gamma(t)}{dt} \cdot \frac{\partial h(\Gamma(t), x(t))}{\partial \Gamma(t)} dt , \quad (12)$$

$$\begin{aligned} W - E &= H(\Gamma(\theta), x(\theta)) - H(\Gamma(\theta), x_0) = \\ &= -h(\Gamma(\theta), x(\theta)) \end{aligned} \quad (13)$$

It is clear from these expressions that the difference between W and E disappears only in special case $x(\theta) = x_0$, and then JE and BKE “reduce” one to another. But in general none of them reduces to other.

It is seen also that E can be interpreted as energy dissipated by the system, and this interpretation of E is valid, along formulae (12) and (13), also at $x(0) \neq x_0$ (and in presence of any other jumps of forces).

E. BKE for open systems, or when JE is out of work

What of the two equalities, JE and BKE, is more useful? The answer depends on character of action of the forces onto system. In reality, one of forces do work at system only when they are changing with time, while another can do even when being constant (if only they are not “equal to zero”). The JE is more convenient in the first case, which demands forces to vary smoothly in theory, since in practice their sharp variations may be non-realizable. Example of such force (parameter) is given e.g. by position of the piston at Fig.1 in [1] (though, its pulling out can be arbitrary fast). What is for the BKE, it is more convenient in the second case, when theoretical jumps (“switchings on” and “switchings off”) of forces are quite adequate to practical actions. Example of such force is given by the current-induced torque at Fig.3 in [1].

When considering the second case, it is natural to establish reference point of forces in (10) to be $x_0 = 0$, thus presuming (without loss of generality) that just at $x = 0$ our system is free from external influences. For example, if $x(t)$ represents potential drop through an electric conductor, or electric force (field) acting onto a particle (charge carrier) in an unbounded (in the “thermodynamic limit”) medium.

At $x \neq 0$, such a system comes to (quasi-) stationary non-equilibrium state with constant dissipation of energy. In other words, such system (or some its part) behaves as non-equilibrium “open system”.

Now, we would like to demonstrate that JE becomes inadequate in such situations. Let the force be sharply “switched on” at initial time moment, i.e. $x(t) = 0$ for $t < 0$ and $x(t) = x = \text{const} \neq 0$ for $t > 0$. Then, as one can see, the quantity W (11) as a function of θ degenerates into constant $W = H(\Gamma, x) - H(\Gamma, 0)$, since at $t > 0$ total system’s energy stays constant.

Consequently, JE (3) gives no information about what happens during non-equilibrium processes at $t > 0$. Moreover, JE loses significance at all, since the average there, $\langle \exp(-W/T) \rangle = \int \exp\{[F(0) - H(\Gamma, x)]/T\} d\Gamma$, turns to infinity. For instance, if denoting with Q ($Q \in \Gamma$) coordinate of particle in the second of mentioned examples, we have $H(\Gamma, x) = H_0(\Gamma) - xQ$. This evidently implies exponential divergency of integration over Q , so that $\Delta F = -\infty$ at $x \neq 0$.

At the same time, $E = h(\Gamma(\theta), x) - h(\Gamma, x) = x[Q(\theta) - Q(0)]$ gives energy transferred during all the observation time from external force’s source to system’s internal degrees of freedom (to thermostat), i.e. dissipated energy. Therefore, the BKE (7), along with other FDR (see Sec.III), brings [12–15] useful information about interrelations of dissipative and fluctuational characteristics of non-equilibrium open system (while its free energy falls out of play).

In this kind of applications it is reasonable, if not necessary, to specify the interaction Hamiltonian (10) in (9) as bilinear form $h(\Gamma, x) = x \cdot Q(\Gamma)$, that is write

$$H(\Gamma, x) = H_0(\Gamma) - x \cdot Q(\Gamma), \quad (14)$$

where $Q(\Gamma)$ are internal variables “conjugated” with external forces. Then the dissipated energy (12) acquires visual (and customary) expression

$$E = \int_0^\theta I(t) \cdot x(t) dt, \quad (15)$$

where $I(t) = dQ(\Gamma(t))/dt$ are “currents”, or “flows”, conjugated with forces.

F. On experimental verification of exact theoretical equalities

Here, we cannot but notice that if authors of the concerned in [1] experiments with torsion pendulum [40]

were aware of our works [11] or [14] then they would have motives to verify, in addition to JE (3), also BKE (7), by measuring not only integral $W = -\int_0^\theta \dot{M} \Theta dt$ but also integral $E = \int_0^\theta M \dot{\Theta} dt$ (dissipated energy).

Simultaneously, it would be possible to verify equality

$$P(E; x) \exp(-E/T) = P(-E; \tilde{x}), \quad (16)$$

in which $P(E; x)$ is probability density distribution of E under given protocol $x = x(t)$, and $\tilde{x}(t) = x(\theta - t)$. This is direct consequence of FDR for probability functionals (see below p.III.B and p.III.F), with $x(t) = M(t)$ (M is the torque) and $Q = \Theta$ (angle of rotation). Of course, the equality (16) as well is valid without any restrictions on time dependence of forces (i.e. torque here).

By the way, notice that experimental “verification” of formally exact inner facts of theory means, in essence, that of either paradigm of the theory as the whole (statistical mechanics) or correctness of posing and description (“stochastic model”) of the experiment itself.

G. Exact equalities for evolution from non-equilibrium states

Let us consider a system with Hamiltonian of type (14) and statistical ensemble determined by non-equilibrium (normalized) initial distribution as follows,

$$D(\Gamma; 0) = \exp\{\beta F(X) - [\beta H_0(\Gamma) + X \cdot Q(\Gamma)]\}, \quad (17)$$

where $\beta = 1/T$, $Q(\Gamma)$ is some set of individual or collective variables of the system, and X is conjugated set of its internal “thermodynamic forces” characterizing its nonequilibrium. Such the distribution corresponds to maximum informational entropy of the system under given mean values of $Q(\Gamma)$ and its internal energy $H_0(\Gamma)$. Application of the abstract equality (7) to this situation, - by replacing $H_0(\Gamma)$ in (7) with $H_0(\Gamma) + TX \cdot Q(\Gamma)$, - yields

$$\langle \exp(-\Delta S) \rangle = 1, \quad (18)$$

$$\begin{aligned} \Delta S &= \beta E + X \cdot [Q(\Gamma(\theta)) - Q(\Gamma)] = \\ &= \int_0^\theta I(t) \cdot [\beta x(t) + X] dt \end{aligned} \quad (19)$$

In concrete examples the quantity ΔS usually can be treated as increment of entropy of the system during its evolution caused by both the initial nonequilibrium and external influences.

In [13–16] it was shown that FDR corresponding to such ensembles create reliable ground for non-linear non-equilibrium thermodynamics.

H. Thermodynamical inequalities

It is well known that for any random quantity A the inequality $\langle \exp A \rangle \geq \exp \langle A \rangle$ is valid, which turns to equality only for $A = \text{const}$.

Substituting $-E$ in place of A , with E from (5) or (12) or (15), and combining this inequality with equality (7), it is not hard to conclude that $\langle E \rangle \geq 0$. Hence, if a system is initially equilibrium, then on average (ensemble average) it always takes from sources of external forces and dissipates a positive amount of energy.

Now, let our system is non-equilibrium already in the beginning of observation, like in the preceding paragraph. Then equality (18) implies inequality $\langle \Delta S \rangle \geq 0$. It allows negative values of the average $\langle E \rangle$, that is now the system can do (useful) work against external forces, when $-\langle E \rangle > 0$. At that, the inequality $\langle \Delta S \rangle \geq 0$ together with (19) establishes restriction on possible value of this work, dependently on the initial system's nonequilibrium.

Next, we shall go to more informative exact statistical equalities including time-reversed process, and consider how results of our works [11, 12, 14] relate to results (in particular, by Crooks [34–36]) what appeared after 1997.

III. TIME REVERSIBILITY OF MICROSCOPIC MOTION AND GENERALIZED FLUCTUATION-DISSIPATION RELATIONS (FDR)

A. Rules of time reversal

For beginning, let us introduce designation $\bar{\Gamma} = \{q, -p\}$ and assume that $H(\bar{\Gamma}, x) = H(\Gamma, \epsilon x)$, where for some of forces (“even”) $\epsilon = 1$ while for all others (“odd”) $\epsilon = -1$. In fact, it always is so, if one suitably defines the forces x and includes into their list all static odd parameters of Hamiltonian when such parameters exist (the standard example is static external magnetic field).

The time reversibility means that any system's phase trajectory can be passed back if making at some its “end” time point, θ , replacement of $\Gamma(\theta)$ by $\tilde{\Gamma} = \bar{\Gamma}(\theta)$ and $x(t)$ by $\tilde{x}(t) = \epsilon x(\theta - t)$. Here $\tilde{\Gamma}$ is initial microstate of backward motion, and t is new time count off back from the old end point.

At that, in the reversed time, the reversed trajectory $\tilde{\Gamma}(t) = \bar{\Gamma}(\theta - t)$ depends on $\tilde{\Gamma}$ and $\tilde{x}(t)$ in exactly the same way as original “direct” trajectory $\Gamma(t)$ depends on Γ and $x(t)$. Correspondingly, observation of any variable $Q(\Gamma)$ with definite time parity, - $Q(\bar{\Gamma}) = \varepsilon Q(\Gamma)$ ($\varepsilon = \pm 1$), - instead of $Q(t) = Q(\Gamma(t))$ on direct trajectory will display $\tilde{Q}(t) = Q(\bar{\Gamma}(\theta - t)) = \varepsilon Q(\theta - t)$ on the reversed trajectory. For the “currents” $I(t) = dQ(t)/dt$ then one will see $\tilde{I}(t) = -\varepsilon I(\theta - t)$ instead of $I(t)$.

B. Generalized FDR

Let $\Phi\{\Gamma(t)\}$ be some (arbitrary) functional of system's phase trajectory on time interval $(0, \theta)$. Consider its

mean value in ensemble of trajectories generated by initial distribution (6):

$$\langle \Phi\{\Gamma(t)\} \rangle_{x(t)} = \int \Phi\{\Gamma(t)\} D_{in}(\Gamma) d\Gamma, \quad (20)$$

where the comment under right angle bracket reminds of external conditions influencing the system's evolution. Along with (20) consider ensemble average $\langle \Phi\{\Gamma(t)\} \exp(-E/T) \rangle_{x(t)}$. In the phase space integral in this average let us (i) use, like above in paragraph II.C, the identity (8), (ii) go from integration variables Γ to $\Gamma(\theta)$, applying the Liouville theorem, then (iii) go from $\Gamma(\theta)$ to $\tilde{\Gamma} = \bar{\Gamma}(\theta)$, and finally (iv) apply the above “rules”. As the result, we come to equality

$$\langle \Phi\{\Gamma(t)\} \exp(-E/T) \rangle_{x(t)} = \langle \Phi\{\tilde{\Gamma}(t)\} \rangle_{\tilde{x}(t)} \quad (21)$$

Notice that due to the operation (iii) on the right here the averaging is performed over initial distribution $D_{in}(\tilde{\Gamma}) = D_{eq}(\tilde{\Gamma}, x_0) = D_{eq}(\Gamma, \tilde{x}_0)$ ($\tilde{x}_0 = \epsilon x_0$) which essentially (and at $x_0 = 0$ or $\epsilon = 1$ literally) does not differs from distribution $D_{in}(\Gamma)$ on the left. In order to underline this principal circumstance, the left and right protocols in (21) can be written as $\{x_0, x(t)\}$ and $\{\tilde{x}_0, \tilde{x}(t)\}$ (thus indicating also preparation of initial states). Below in this paragraph we suppose, as in [11], that $x_0 = 0$.

Choosing $\Phi\{\Gamma(t)\}$ properly, it is easy to transform (21) into various “generalized FDR” for characteristic or probabilistic functionals. For instance, for some set of variables $Q(\Gamma(t))$ let

$$\begin{aligned} \Phi\{\Gamma(t)\} &= \prod_{0 \leq t \leq \theta} \delta[Q(t) - Q(\Gamma(t))] \equiv \\ &\equiv P\{Q(t); x(t), \Gamma\} \end{aligned} \quad (22)$$

This is probabilistic functional (probability density distribution in functional space) of trajectories $Q(\Gamma(t))$ which corresponds to concrete initial microstate Γ . Then the average (20),

$$P[Q; x] = \int P\{Q; x, \Gamma\} D_{in}(\Gamma) d\Gamma, \quad (23)$$

is probabilistic functional (PF) of trajectories $Q(\Gamma(t))$ in the framework of given statistical ensemble of phase trajectories.

Further, let the set $Q(\Gamma)$ is wide enough to express in its terms the functional E : $E = E[Q; x]$. Then insertion of (22) into (21), subject to (23), leads to formula (7) from [11],

$$P[Q; x] \exp(-E/T) = P[\tilde{Q}; \tilde{x}] \quad (24)$$

(in contrast to [11], here the parities ϵ, ε are included by the symbol “tilde”). Both sides of (24) relate to one and the same (equilibrium) initial distribution $D_{eq}(\Gamma, 0)$.

In case of “bilinear” Hamiltonians (14) the set Q what appears in (24) should cover the set appearing in (14).

Therefore the second can be identified with the first if setting all additional conjugated forces to be identically equal to zero.

Notice, besides, that at $\Phi\{\Gamma(t)\} = \exp\{-[W - E]/T\}$ and $x(0) = x_0$ relation (21) turns to the JE (3).

In order to generalize equalities (21) and (24) to the “initially non-equilibrium” ensemble introduced in p.II.G, it is sufficient to replace E by ΔS and add the thermodynamic forces X to comments under angle brackets. The resulting analogue of (24) can be represented by formula (1) from Introduction ((2) from [17]), where $\Pi_+ = \{\Pi; x, X\}$ symbolizes arbitrary “direct” process Π together with conditions of its observation (external forces $x(t)$ and thermodynamical forces X), $\Pi_- = \{\tilde{\Pi}; \tilde{x}, \tilde{X}\}$ is time-reversed process in respect to Π_+ , $\Delta S = \Delta S(\Pi_+)$ is system’s entropy increment in the direct process, and $\tilde{X} = \epsilon X$. Of course, according to (19), $\Delta S(\Pi_-) = -\Delta S(\Pi_+)$.

C. From old to new relations

It was already pointed out that left and right sides of equalities (21) and (24) describe evolutions of a system under consideration from one and the same equilibrium state, - more precisely, $D_{eq}(\Gamma, x_0)$ (on the left) and $D_{eq}(\bar{\Gamma}, x_0)$ (on the right), - under mutually time-reversed protocols $x(t)$ and $\epsilon x(\theta - t)$. Obviously, if at that $x(\theta) \neq x(0)$ then at least on one side the forces have a jump in the beginning of observation.

This consequence of the reversibility is just what should be used if the system has no equilibrium states at $x \neq x_0$ or has but we are interested in processes of transition to such states from $x = x_0$ (see p.II.E). Otherwise, it is possible to replace (21) and (24) by formally equivalent relations like “Crooks equalities” [34–38] where the mentioned jumps disappear since these relations do compare evolutions from different equilibrium states, $D_{eq}(\Gamma, x(0))$ and $D_{eq}(\bar{\Gamma}, x(\theta))$.

To equal out such the relations, let us make in (21) replacement

$$\Phi\{\Gamma(t)\} \Rightarrow \Phi\{\Gamma(t)\} \times \exp\{-[H(\Gamma(\theta), x(\theta)) - H(\Gamma(\theta), x_0)]/T\}, \quad (25)$$

where $\Phi\{\Gamma(t)\}$ again is arbitrary functional. Then on the left in place of $\exp(-E/T)$ another exponential will appear, $\exp(-W/T)$, while on the right the averaging over initial microstates will be performed with another weight function,

$$\exp\{[F_0 - H(\Gamma, \tilde{x}(0))]/T\} = \exp(-\Delta F/T) D_{eq}(\Gamma, \epsilon x(\theta))$$

(we took into account that $F_0 = F(x_0)$ and $F(\epsilon x) = F(x)$). Choosing here $x(0) = x_0$, we obtain relation

$$\begin{aligned} \langle \Phi\{\Gamma(t)\} \exp(-W/T) \rangle_{x(t)} &= \\ &= \exp(-\Delta F/T) \langle \Phi\{\tilde{\Gamma}(t)\} \rangle_{\tilde{x}(t)} \end{aligned} \quad (26)$$

Now left and right sides correspond to protocols $x(t)$ and $\tilde{x}(t)$ (or, if indicating also initial states, $\{x(0), x(t)\}$ and $\{\tilde{x}(0), \tilde{x}(t)\}$) which continuously join one with another. This is achieved thanks to difference between (equilibrium) initial ensembles on the left and on the right.

Next, insertion of (22) into (26) yields equality

$$P[Q; x] \exp(-W/T) = \exp(-\Delta F/T) P[\tilde{Q}; \tilde{x}], \quad (27)$$

replacing (24) after the transition to new ensembles. This equality can be deduced also directly from (24) with the help of multiplying (24) by the exponential factor from (25) and exploiting the expansion (22).

D. More about interrelations of “new” and “old” results

It should be emphasized once again that the demonstrated transformation of “old” relations (FDR) (suggested in 1977-1979) to “new” ones (suggested in 1997-1999) presumes that systems under consideration are closed, i.e. possess thermodynamically equilibrium states $D_{eq}(\Gamma, x)$ at any (constant) force values x . For open systems, - as we saw in p.II.E, - distributions $D_{eq}(\Gamma, x)$ do not exist (are non-normalizable) at $x \neq 0$. Consequently, all “new” relations, (26), (27), etc., where W and/or ΔF appear, lose significance, like JE (3) does. At the same time, “old” relations do work with success.

Especially it is necessary to dwell on the Markovian realization of FDR. As it was shown in [11], generated FDR are formally compatible with assumption about Markovian property of evolution and fluctuations of one or another set of variables $Q(\Gamma(t))$, though really the “Markovianity” never takes place (if only Q does not coincide with Γ). This fact allows to formulate definite rules of construction of such Markovian “stochastic models” what satisfy all FDR and therefore automatically agree with both reversibility of microdynamics and principles of (statistical) thermodynamics of irreversible processes [11, 12, 14, 15, 18, 19].

In respect to closed systems, such rules in the main were considered already by Stratonovich (see [10] and references therein and in [11, 12, 18]) who had started from the principle of detailed balance. We in [11] extended his results to systems with time-dependent external forces.

The corresponding theory (Stratonovich’s and our) presented in 1977, if not earlier, completely include the Crooks theory which appeared first in Markovian representation [34, 35]. Differences between these theories are in non-principal details only: terminology, mathematical design and notations. At that, transition from Markovian language to general one formally adds nothing new, since deterministic (Hamiltonian) dynamics can be treated as limit case of Markovian random process (with $Q = \Gamma$).

In 1978-1981 we formulated generalization of the Stratonovich’s Markovian theory to open systems [12, 14, 15, 18, 19]. The corresponding our results, as far as we know, still are not reproduced by new authors.

Below, we shall speak about open systems. But before, since referring to Markovian models, we have to discuss them at least briefly.

E. Markovian FDR

Let a set of variables $\psi(t) = \psi(\Gamma(t))$ is sufficient to express through it the interaction energy (10) (for closed systems) and/or the dissipated power (for open systems), that is the integrand in (12). If $\psi(t)$ is assumed to be a Markovian random process, then complete information about it is concentrated in kinetic equation for density $P(t, \psi)$ of its probability distribution:

$$\dot{P}(t, \psi) = \mathbf{K}(\psi, \nabla, x(t)) P(t, \psi), \quad (28)$$

where $\nabla = \partial/\partial\psi$, and \mathbf{K} is differential or integral “kinetic operator” (as, for instance, in the Fokker-Planck equation and Kolmogorov equation, respectively). For a closed system, this operator should satisfy operator-valued equality

$$\mathbf{K}(\psi, \nabla, x) P_{eq}(\psi; x) = P_{eq}(\psi; x) \mathbf{K}^\dagger(\varepsilon\psi, \varepsilon\nabla, \varepsilon x) \quad (29)$$

which accumulates all the consequences of FDR. Here, \dagger is symbol of operator conjugation, or transposing, in the Sturm-Liouville sense, and $P_{eq}(\psi; x) = \int \delta(\psi - \psi(\Gamma)) D_{eq}(\Gamma, x) d\Gamma = P_{eq}(\varepsilon\psi; \varepsilon x)$ is equilibrium distribution under constant forces. It represents stationary solution of (28).

If a system is open in respect to forces x , then (29) should be replaced by another operator-valued equality,

$$\begin{aligned} \mathbf{K}(\psi, \nabla, x) P_{eq}(\psi) &= \\ &= P_{eq}(\psi) [\mathbf{K}^\dagger(\varepsilon\psi, \varepsilon\nabla, \varepsilon x), +\mathbf{N}(\psi, x)/T], \end{aligned} \quad (30)$$

where $P_{eq}(\psi) = P_{eq}(\psi, 0) = P_{eq}(\varepsilon\psi)$ is equilibrium distribution (existing at $x = 0$ only), and $\mathbf{N}(\psi, x)$ is the dissipated power ($x \cdot I(\psi)$ in the case (14)-(15)). Now, stationary solution of (28) at $x \neq 0$ gives non-equilibrium distribution of ψ .

If there are simultaneously forces of both “closed type” and “open type”, then instead of equalities (29) and (30) one should write their obvious hybrid.

Notice that (29) is merely time-local form of the detailed balance (of direct and reversed processes of $\psi(t)$ ’s interaction with thermostat), while (30) expresses violation of the detailed balance to the extent of instant “entropy production” $N(\psi, x)/T$.

It is quite evident that equality (30) represents differential form of the equality (24) (with $Q = \psi$). Less evident fact is that equality (29) represents differential form of (27), i.e. “Crooks equality”. Nevertheless, it is so, and one can say that Crooks equality is expression of time-local detailed balance which (as shown in [11]) takes place even in strongly disturbed closed systems.

It is necessary also to pay attention to degenerated Markovian models where $\mathbf{K}(\psi, \nabla, x) = -\nabla \cdot \mathbf{K}(\psi, x)$.

This means, obviously, that evolution of $\psi(t)$ is deterministic although may be dissipative, “with friction” (dependent on system’s state), that is thermostat does not produce a noise (“has zero temperature”). At that, the distributions (“invariant measures”) $P_{eq}(\psi, x)$ become singular, while limit form of the relations (29) and (30) (in the zero noise limit) prescribes inversion of sign of the friction under time reversal. Therefore all FDR remain valid (at least in definite class of trajectories $\psi(t)$). Such models help to describe some numeric experiments [41] which, however, are very artificial and far from actual problems of statistical mechanics.

F. Fluctuation theorems

Let us return to p.III.B and choose $\Phi\{\Gamma(t)\} = \delta(E - E\{\Gamma(t), x(t)\})$, where $E\{\Gamma(t), x(t)\}$ is dissipated energy, (12) or (15), considered as functional of phase trajectory of a system and applied forces (eventually, function of initial point Γ and functional of $x(t)$). Then average (20) turns to density $P(E; x)$ of probability distribution of E , and relation (21) gives us above mentioned equality (16).

The same follows from (24) and clear expression $P(E; x) = \int \delta(E - E[Q, x]) P[Q; x] dQ$, if notice that $E[\tilde{Q}, \tilde{x}] = -E[Q, x]$. In quite similar way FDR (1) implies analogue of (16) for distribution of ΔS :

$$P(\Delta S; x, X) \exp(-\Delta S) = P(-\Delta S; \tilde{x}, \tilde{X}) \quad (31)$$

Such particular FDR, transparently visible even without literal manipulations, in current literature are termed “fluctuation theorems” (FT) [34, 37–39]. We in our time had written out and exploited them in somehow different forms, more practical at that time (see below).

Further, we shall illustrate, by the examples from p.II.E, how FT were applied by us in 1979 [12] and later to open systems.

G. FDR for transport processes

Consider, for demonstration of one of possible applications of FDR, charge transport through a conductor under constant (after switching on at $t = 0$) voltage drop x . Hamiltonian of the system has form (14) where $Q(\Gamma)$ represents transported charge. We are interested in amount of transport during the observation time interval: $\Delta Q(\theta) = Q(\Gamma(\theta)) - Q(\Gamma)$. The corresponding dissipated energy equals to $E = x\Delta Q$.

Let us combine exact FDR and a simple stochastic model of the system. The FDR will be delegated by relation (FT) (16) written in the form

$$P(\Delta Q; x) \exp(-x\Delta Q/T) = P(-\Delta Q; x), \quad (32)$$

in fact used in [12], with probability density P now relating to charge. What is for the model, assume that

our conductor is a contact (like e.g. $p - n$ -junction) and therefore charge is transported through it by discrete portions $\pm e$ forming two opposite Poissonian random flows. This means that average value of the current, $\bar{I}(x) = \langle \Delta Q \rangle / \theta$, and spectral power density of the current noise, $S(x) = [\langle \Delta Q^2 \rangle - \langle \Delta Q \rangle^2] / \theta$, are expressed by formulae

$$\bar{I} = e[n_+ - n_-], \quad S = e^2[n_+ + n_-], \quad (33)$$

in which $n_{\pm} = n_{\pm}(x)$ are mean numbers of the elementary charge portions transferred per unit time in opposite directions.

Clearly, FDR establish definite connection between $n_+(x)$ and $n_-(x)$. In [12] it was extracted from relations for characteristic function of ΔQ equivalent to (32). Instead of this, here we merely can surmise that relation (32) is valid not only in respect to ΔQ as the whole but also in respect to elementary transfer events:

$$n_+(x) \exp(-ex/T) = n_-(x) \quad (34)$$

From this relation and (33) one obtains following relation between power of on-equilibrium noise and mean current (current-voltage characteristics):

$$S(x) = e\bar{I}(x) \coth(ex/2T) \quad (35)$$

At $e|x| \ll 2T$ it reduces to the Nyquist formula for “thermal noise” while in the opposite case to the formula for “shot noise”.

In such way FDR help to reveal universal connections between dissipative non-linearity of a transport process ($I(x)$), its noise characteristics ($S(x)$) and type of its statistics (here, simple Poissonian one). More complicated examples of this kind and corresponding more general FDR can be found in [12–15, 17, 23–25].

H. FDR and 1/f-noise

Just considered stochastic model has principal defect: in it the elementary random events have *a priori* (“in advance”) prescribed relative frequency (time-averaged number of events per unit time, or their “probability per unit time”), $n_{\pm}(x)$, independent on concrete experiment, i.e. concrete phase trajectory of a system. Although, as it was shown by Krylov many years ago [42], statistical mechanics gives no grounds for such assumptions.

One can understand this statement already on intuitive level. Indeed, the mentioned assumption would be likely if the system remembered a number of past events and compensated its deviations from a “norm” by means of opposite deviation of number of next events. But this is impossible if the system forgets about events soon after they happened. Then it does not distinguish between “norm” and “deviation” and therefore produces fluctuations in the number of events proportionally to its “normal” (average) value. This means that relative frequency

of events (“probability per unit time”) undergoes low-frequency fluctuations with 1/f-type spectrum.

For the first time similar reasonings were suggested and mathematically formulated in [43] and [21, 22] and later confirmed on the base of statistical mechanics in [23–25, 28, 44, 45] and other works, first of all in application to random walks (“Brownian motion”) of atomic-size particles.

It should be underlined that the fluctuations (“1/f-noise”) of relative frequencies by their very nature do not violate existing (anyway prevalent) balance or definite disbalance of mutually time-reversed events (processes). Therefore, - as it follows from generalized FDR [21–23, 25], - various particular FDR like (34) hold also for fluctuating relative frequencies and all derived “kinetic” quantities. For example, the Einstein relation $D = T\mu$ between diffusivity and mobility of a walking particle, D and μ , can be extended to their fluctuations [23] (as well as the Nyquist formula to fluctuations of conductance and fluctuations of “instant” spectral power density of “white” electric noise [21, 25]). At that, to substantiate such statements, the relation (32) (formula (A4) from [23]) is quite sufficient.

Due to these circumstances, - as it was suggested already in [43], - one can separate fast fluctuations (white noise) and low-frequency fluctuations (1/f-noise), making use of primitive phenomenological language but taking in mind its rigorous statistical-mechanical equivalent. Next, consider in such way statistics of random walk of probe (“marked”) gas particle, returning to the second example from p.II.E and basing on results of [23, 28, 44, 45].

I. Molecular Brownian motion and FDR

Now, in the relation (32) ((A4) from [23]) ΔQ will denote displacement, or path, of “Brownian particle” (BP). Let R be projection of ΔQ onto the external force direction. In the widely known simplest stochastic model of Brownian motion the particular FDR (FT) (32) is satisfied by the Gaussian distribution

$$P(R; x) = P_{\mu}(R; x) \equiv \frac{\exp[-(R - \mu x \theta)^2 / 4T\mu\theta]}{\sqrt{4\pi T\mu\theta}} \quad (36)$$

Here, it is assumed, of course, that the observation time θ is much greater than velocity relaxation time (mean free path time), τ , of BP (probe gas atom).

However, honest consideration of the exact BBGKY equations for infinite chain of many-particle distribution functions of a fluid shows that expression (36) is incompatible with absence of (instant) statistical correlations between the BP and gas atoms far distanced from it. A true expression (first obtained in [44] and then by different method in [28, 45]) can be represented by superposition of the Gaussian distributions with various values of BP’s mobility:

$$P(R; x) = \int_0^\infty P_\mu(R; x) U_\theta(\mu) d\mu, \quad (37)$$

$$U_\theta(\mu) = \frac{\mu_0^2}{\mu^3} \exp\left(-\frac{\mu_0}{\mu}\right) \Xi\left(\frac{T\mu}{v_0^2\theta}\right), \quad (38)$$

where $\Xi(\cdot)$ is a “cut off” function which fast decreases at infinity and equals to unit at zero, $\Xi(0) = 1$, and v_0 is characteristic thermal velocity of gas atoms (speed of sound). From here for variances of the BP’s path and the dissipated energy $E = xR$ it follows that

$$\langle R, R \rangle = 2T\mu_0\theta + (\mu_0 x \theta)^2 F\left(\ln \frac{\theta}{\tau}\right), \quad (39)$$

$$F(z) \approx z, \quad (40)$$

Here and below the angle brackets with commas inside them (Malakhov’s cumulant brackets) denote joint cumulant of random quantities separated by the commas. The second term in (39) and (40) says about 1/f-fluctuations of mobility and dissipated power, respectively. At $x = 0$, similar asymptotic characterizes fourth-order cumulant of R , thus reflecting identical fluctuations of BP’s diffusivity $D = T\mu$ [21, 25, 43].

The function $U_\theta(\mu)$ (38) is effective BP’s mobility probability distribution. Its power-law long tail is generally typical for distributions accompanying 1/f-noise [22, 43]. At $\langle E \rangle = \mu_0 x^2 \theta \gtrsim T$ this cubic tail manifests itself in the path distribution (37) on the right (if $x > 0$): $P(R; x) \approx \langle R \rangle^2 / R^3$ at $R > \langle R \rangle$. Correspondingly, the same tail appears in distribution of dissipated energy in (16): $P(E; x) \approx \langle E \rangle^2 / E^3$ at $E > \langle E \rangle$. Hence, probabilities of “large deviations” of the path and dissipated energy highly exceed that predicted by the Gaussian model (36).

Such distributions were repeatedly observed in experiments with non-stationary photo-currents (charge injection currents) [46].

It is interesting that most short way to these results runs from the FDR [28]. Let us choose in (21)

$$\Phi\{\Gamma(t)\} = \delta(Q(\theta) - R) \delta(Q) \prod_k [1 + \phi(q_k)],$$

where $\phi(q)$ is some function of space coordinates, and the product is taken over all gas atoms (except the BP itself). Then, left side of (21) characterizes influence of initial spatial non-uniformity of gas onto BP’s walk, while right-hand side describes statistical correlations between BP’s path (during all the observation time) and current microstate of gas (in configurational space). Further, choosing the function $\phi(q)$ properly, one can extract from (21) relation

$$\nu \frac{\partial P(R; x)}{\partial \nu} = P(R; x) \int [\nu(\rho|R; x) - \nu] d^3\rho, \quad (41)$$

where ν is mean density of gas, $\nu(\rho|R; x)$ is conditional average value of gas density at distance ρ from BP under given its path, and for simplicity gas is assumed rarefied enough (besides, our designations here somehow differ from that in [28]).

From (41) it follows that

$$\partial \ln P(R; x) / \partial \ln \nu > -\nu \Omega, \quad (42)$$

where Ω represents characteristic space volume to which the correlations of gas with BP do extend. On the other hand, in the Gaussian model (36), in view of the known dependence $\mu \propto D \propto 1/\nu$, we have

$$\frac{\partial \ln P_\mu(R; x)}{\partial \ln \nu} = \frac{1}{2} - \frac{\langle E \rangle}{4T} \left[\left(\frac{R}{\langle R \rangle} \right)^2 - 1 \right]$$

Comparing this expression with inequality (42), we see that they certainly are incompatible if the “correlation volume” Ω is bounded above by a finite value.

Hence, if the gas stays indifferent to (forgets about) outcome, R , of BP’s walk, then it is unable to suppress large values of R so categorically as the law (36) does require.

At the same time, the law (37)-(38) is well compatible with (42), at $\Omega = 2/\nu$. Of course, this value of Ω can not be obtained from FDR themselves only, its calculation needs in the whole BBGKY hierarchy [23, 44].

J. Variance of dissipation fluctuations

The previous paragraph gave example of large fluctuations of dissipation whose magnitude, according to (40), is of order of mean dissipation value. Another such example was considered in [24]. If energy is dissipated through not one but many, $N \gg 1$, degrees of freedom, then variance of dissipation fluctuations, along with magnitude of power-law tail of their distribution, will be approximately N times smaller. Anyway, for systems with Hamiltonians of type (14) there are exact equalities [24]

$$\langle E \rangle = \frac{1}{T} \int_{1>2} x(1) \langle I(1), I(2) \rangle_{x(t)\eta(t-2)} x(2) d1d2, \quad (43)$$

$$\langle E, E \rangle = 2T \langle E \rangle + \frac{2}{T} \int_{1>2>3} x(1) x(2) \times \\ \times \langle I(1), I(2), I(3) \rangle_{x(t)\eta(t-3)} x(3) d1d2d3 \quad (44)$$

Here, the ciphers replace time arguments (and their numbers), and $\eta(t)$ is the Heavyside step function, that is in second- and third-order cumulants in the integrands the most early (right-hand) currents ($I(2)$ and $I(3)$, respectively) belong to still undisturbed (equilibrium) system. These equalities follow from general FDR for cumulants [12, 14] (importantly, in [14] we had written out two variants of such FDR, for systems of “open” type and “closed” type separately).

In many applications one can suppose that the third-order cumulant (or result of its integration) in (44) vanishes at $x = 0$. Then, under weak disturbance the first term in (44) is $\propto x^2$ while the second $\propto x^4$, and

$$\langle E, E \rangle = 2T \langle E \rangle + \frac{2}{T} \int_{\substack{1>4 \\ 2>3,4}} x(1)x(2) \times \\ \times \langle I(1), G(2, 3), I(4) \rangle_0 x(3)x(4) d1d2d3d4, \quad (45)$$

where $G(2, 3) = [\delta I(2)/\delta x(3)]_{x=0}$ is dynamical (random from the ensemble viewpoint) linear response of the currents to forces [24]. However, the second term does not reduce to a small correction of the first, if it grows with observation time faster than linearly. For instance, when it grows, like in (40), approximately $\propto \theta^2$. At that, the integral in (45) determines values and spectra of equilibrium 1/f-fluctuations in kinetic coefficients of the system and power dissipated by it.

IV. CONCLUSION

We have presented to readers a brief review of the “generalized fluctuation-dissipation relations (theorems)”, for the first time obtained in 1977 in our work [11], in comparison with much later analogous results, in particular, the “fluctuation theorems”. Both our “old” and later “new” results express important consequences of time symmetry (reversibility) of microscopic motion, substantially determining statistical and dissipative properties of thermodynamically non-equilibrium physical systems. However, the expounded comparison gives suffi-

cient grounds to state, - in contrast to the misunderstandings observed in current literature (see Introduction), - that “new” results do not contain a principal novelty and in the sense of generality are significantly inferior to “old” results (if not per se reproducing them or even earlier results by Stratonovich).

The matter is that our approach (reflected in the present paper), - based on constructive play with statistical ensembles and Hamiltonians, - is more flexible and fruitful. Due to it, our “generalized fluctuation-dissipation relations” (FDR) from the very beginning were represented in the form foreseeing their applications to systems of not closed type only but also open type, while “new” results were oriented to the first of these two and lose meaning in respect to the second.

In the framework of our “old approach” any “new result” can be obtained from the “old” FDR, but the opposite in the framework of “new approach” is in general impossible. Nevertheless, some of new formulations of FDR indeed are more comfortable in their domains of applicability and thus enlarge collection of practical applications of FDR.

On the whole, the generalized FDR bring all necessary tools for construction of thermodynamically correct stochastic models of real processes and systems. Regardless of degree of complexity or roughness of a model, observance of FDR at its level ensures its qualitative agreement with rigorous statements of statistical mechanics (and sometimes even leads closely to quantitative agreement, as was demonstrated, in particular, by examples in last paragraphs of the present paper). Hardly this useful potential of FGR some day will be exhausted.

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- [1] Pitaevski L P Phys.Usp. **54** 625 (2011) [Pitaevskii L.P. UFN **181** 647 (2011)]
 - [2] Landau L D and Lifshitz E M Statistical physics. P.I (Butterworth-Heinemann 1980) [Statisticheskaya fizika. Ch.I (M. Fizmatlit 2004)]
 - [3] Lifshitz E M and Pitaevski L P Physical kinetics (Pergamon Press 1981) [Fizicheskaya kinetika (M. Nauka 1978)]
 - [4] Lifshitz E M and Pitaevski L P Statistical physics. P.II (Butterworth-Heinemann 1980) [Statisticheskaya fizika. Ch.II (M. Nauka 1978)]
 - [5] Einstein A. Dissertation (Zurich University 1905) [Einstein A. Izbrannye nauchnye trudy. Tom III (M. Nauka 1966 stat'ya 6)]
 - [6] Nyquist H. Phys.Rev. **32** 110 (1928)
 - [7] Callen H.B. and Welton T.A. Phys.Rev. **83** 34 (1951)
 - [8] Efremov G.F. ZhETF **51** 156 (1966); ZhETF **54** 2322 (1968) (in Russian; transl. into English in Sov.Phys.-JETP)
 - [9] Stratonovich R.L. ZhETF **58** 1612 (1970) (in Russian; transl. into English in Sov.Phys.-JETP)
 - [10] Stratonovich R.L. Nonlinear Nonequilibrium Thermodynamics. I. Linear and Nonlinear Fluctuation-Dissipation Theorems. II. Advanced Theory (Springer-Verlag 1992 1994) [Stratonovich R.L. Nelineynaya neravnovesnaya termodinamika (M. Nauka 1985)]
 - [11] Bochkov G.N. and Kuzovlev Yu.E. Sov.Phys.-JETP **45** 125 (1977) [ZhETF **72** 238 (1977)] http://www.jetp.ac.ru/cgi-bin/dn/e_045_01_0125.pdf
 - [12] Bochkov G.N. and Kuzovlev Yu.E. Sov.Phys.-JETP **49** 543 (1979) [ZhETF **76** 1071 (1979)] http://www.jetp.ac.ru/cgi-bin/dn/e_049_03_0543.pdf
 - [13] Bochkov G.N. and Kuzovlev Yu.E. Sov.Phys.-JETP **52** No.6 1133 (1980) [ZhETF **79** 2239 (1980)] http://www.jetp.ac.ru/cgi-bin/dn/e_052_06_1133.pdf
 - [14] Bochkov G.N. and Kuzovlev Yu.E. Physica **A106** 443 (1981) [Preprint NIRFI 138 (Gorkii USSR 1980)]
 - [15] Bochkov G.N. and Kuzovlev Yu.E. Physica **A106** 480 (1981) [Preprint NIRFI 139 (Gorkii USSR 1980)]
 - [16] Bochkov G.N. and Kuzovlev Yu.E. JETP Letters **30** No.1 46 (1979) [Pis'ma v ZhERF **30** 52 (1979)] http://www.jetpletters.ac.ru/ps/1361/article_20580.pdf
 - [17] Bochkov G.N. Kuzovlev Yu.E. and Troitskii V.S. Soviet Physics Doklady **29** 458 (1984) [DAN SSSR **276**

- 854 (1984)]
- [18] Bochkov G.N. and Kuzovlev Yu.E. Radiophysics and Quantum Electronics **21** 1019 (1978) [Izv.VUZov.Radiofizika **21** 1468 (1978)]
 - [19] Bochkov G.N. and Kuzovlev Yu.E. Radiophysics and Quantum Electronics **23** 947 (1980) [Izv.VUZov.Radiofizika **23** 1428 (1980)]
 - [20] Bochkov G.N. and Kuzovlev Yu.E. Radiophysics and Quantum Electronics **24** 585 (1981) [Izv.VUZov.Radiofizika **24** 855 (1981)]
 - [21] Bochkov G.N. and Kuzovlev Yu.E. Sov.Phys.-Usp. **26** 829 (1983) [UFN **141** 151 (1983)]
 - [22] Bochkov G.N. and Kuzovlev Yu.E. Radiophysics and Quantum Electronics **27** 811 (1984) [Izv.VUZov.Radiofizika **27** 1151 (1984)]
 - [23] Kuzovlev Yu.E. Sov.Phys.-JETP **67** (12) 2469 (1988) [ZhETF **94** (12) 140 (1988)]
http://www.jetp.ac.ru/cgi-bin/dn/e_067_12_2469.pdf
 Author's translation:
<http://arxiv.org/abs/0907.3475>
 - [24] Kuzovlev Yu.E. JETP **84** 1138 (1997) [ZhETF **111** 2086 (1997)]
 - [25] Kuzovlev Yu.E. arXiv cond-mat/9903350
 - [26] Kuzovlev Yu.E. JETP Letters **78** No.2 92 (2003) [Pis'ma v ZhETF **78** 103 (2003)]
 - [27] Kuzovlev Yu.E. arXiv cond-mat/0501630 cond-mat/0602332
 - [28] Kuzovlev Yu.E. arXiv 0802.0288 0803.0301
 - [29] Kuzovlev Yu.E. arXiv 1106.0589
 - [30] Kuzovlev Yu.E. arXiv 1108.1740
 - [31] Jarzynski . Phys.Rev.Lett. **78** 2690 (1997)
 - [32] Jarzynski C. J.Stat.Phys. **96** 415 (1999)
 - [33] Jarzynski C. Phys.Rev. E **73** 046105 (2006)
 - [34] Crooks G.E. Phys.Rev. E **60** 2721 (1999)
 - [35] Crooks G.E. J.Stat.Phys. **90** 516 (1998)
 - [36] Crooks G.E. Phys.Rev. E **61** 2361 (2000)
 - [37] Esposito M. Harbola U. and Mukamel S. Rev. Mod. Phys. **81** 1665 (2009)
 - [38] Campisi M. Hänggi P. and Talkner P. Rev. Mod. Phys. **83** 771 (2011)
 - [39] Campisi M., Talkner P. and Hänggi P. Phil.Trans.R.Soc. **A 369** 291 (2011) {DOI: 10.1098/rsta.2010.0252}
 - [40] Douarche F. Ciliberto S. Petrosyan A. and Rabbiosi I. Europhys. Lett. **70** 593 (2008)
 - [41] Evans D.J. Cohen E.G.D. and Morriss G.P. Phys. Rev. Lett. **71** 2401 (1993)
 - [42] Krylov N.S. Works on the foundations of statistical physics (Princeton 1979) [Raboty po obosnovaniyu statisticheskoi fiziki (Izd. AN SSSR M.-L. 1950)]
 - [43] Bochkov G.N. and Kuzovlev Yu.E. Pis'ma v ZhTF **8** No.20 1260 (1982) (in Russian transl. to English in Sov.Tech.Phys.Lett.)
 Kuzovlev Yu.E. and Bochkov G.N. Preprint NIRFI No.157 (Gorkii USSR 1982) (in Russian)
 Kuzovlev Yu.E. and Bochkov G.N. Radiophysics and Quantum Electronics **26** 228 (1983) [Izv.VUZov.Radiofizika **26** 310 (1983)]
http://yuk-137.narod.ru/what_is_1_by_F_noise.pdf
 - [44] Kuzovlev Yu.E. arXiv cond-mat/0609515
 - [45] Kuzovlev Yu.E. Theoretical and Mathematical Physics **160** (3) 1301 (2009) {DOI:10.1007/s11232-009-0117-0} [TMF **160** (3) 517 (2009)]
 Kuzovlev Yu.E. arXiv 0908.0274
 - [46] Kuzovlev Yu.E. arXiv 1008.4376